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Proper L–S category, fundamental pro-groups and 2-dimensional proper co-H-spaces

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Abstract

This paper presents a study of one-ended locally finite CW-complexes with proper L–S category $\leqslant 2$. We detect the class of towers of groups which can be the fundamental pro-group of a space of proper L–S category 2. A second part of the paper is concerned with two-dimensional CW-complexes. For these, we give different characterizations of spaces with proper L–S category $\leqslant 2$.

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1. Introduction

The Lusternik–Schnirelmann (L–S) category of a space X is the least number n such that there is an open cover of X consisting of n elements each of which is contractible in X. The homotopical properties of this intricate numerical invariant has drawn considerable interest. The basic work on the homotopical significance of the L–S category is due to Borsuk; Borsuk's work was continued by Fox. Variations on the definition of L–S cat-

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egory and definitions of L–S category type invariants in various topological settings can be found in the literature; see [24] or [21] for a survey on L–S category and [15] for details. In [4,6] numerical invariants of Lusternik–Schnirelmann type were introduced in the proper category of non-compact spaces and proper maps. Moreover, it was proved in [6] that Euclidean n-spaces represent the unique (up to proper homotopy for $n = 3$) open n-manifolds with proper L–S category 2.

As part of the major project of computing the proper L–S category of open one-ended 3-manifolds, we devote this paper to the study of one-ended locally finite CW-complexes with proper L–S category 2. For this, we define the notion of properly based L–S category in order to deal with the subtle question of the "base point" in proper homotopy theory originated by the dependence of the fundamental pro-group on the strong end represented by the base ray. Although different rays lead to different values of the properly based L–S category (Example 3.5), these values can only differ at most by 1 from the absolute proper L–S category (Proposition 3.4). Furthermore, we detect the class of towers of groups which can be the fundamental pro-group of a space of proper category $\leqslant 2$ (Proposition 4.6); namely these towers are coproducts $\underline{L} \vee \underline{P}$ where $\underline{L}$ is a free tower; that is, a tower of the form

$$\underline{L} = \left\{ L_0 \xleftarrow{i_1} L_1 \xleftarrow{i_2} \cdots \right\}$$

where $L_i = \langle B_i \rangle$ are free groups of basis B_i such that $B_{i+1} \subset B_i$, the differences $B_i - B_{i+1}$ are finite and $\bigcap_{i=0}^{\infty} B_i = \emptyset$ and the bonding homomorphisms i_k are induced by the corresponding basis inclusions. Moreover, $\underline{P}$ is a telescopic tower; that is, a tower of the form

$$\underline{P} = \left\{ P_0 \xleftarrow{p_1} P_1 \xleftarrow{p_2} \cdots \right\}$$

where $P_i = \langle D_i \rangle$ are free groups of countable basis D_i such that $D_{i-1} \subset D_i$, the differences $D_i - D_{i-1}$ are finite (possibly empty) and the bonding homomorphisms p_k are the obvious projections.

That result plays a key role in proving that the proper L–S category of a large class of Whitehead manifolds is 4; see [12].

As a by-product of the methods used in the first part of this paper we obtain in a geometrical fashion a purely algebraic result stating that the class of towers of the form $\underline{L} \vee \underline{P}$ is closed under retracts in the category of finitely presented towers of groups (Corollary 5.8).

A second part of the paper is concerned with two-dimensional CW-complexes. For these, we give different characterizations of spaces with proper L–S category $\leqslant 2$; in particular they coincide with the two-dimensional proper co-H-spaces, or equivalently they have the same proper Whitehead 1-type as the "proper Eilenberg–MacLane space" $B(\underline{L} \vee \underline{P})$ obtained by attaching along the half line a "spherical object" $S^1 \times \{k_j\} \cup \mathbb{R}_+$, cylinders $C_j = S^1 \times [n_j, \infty)$ and Euclidean planes $E_j = S^1 \times [m_j, \infty)/S^1 \times \{m_j\}$; here k_j, n_j and m_j are finite (possibly empty) or infinite sequences of distinct natural numbers (Corollary 6.4). Moreover, we prove in Proposition 6.3 that two locally compact one-ended two-dimensional connected CW-complexes X^2 and Y^2 have the same fundamental pro-group if and only if there are spherical objects S^2_γ and S^2_δ such that $X^2 \vee S^2_\gamma \simeq Y^2 \vee S^2_\delta$. This result is the proper analogue of the two-dimensional case of a theorem due to Whitehead which shows that the homotopy types of finite n-dimensional connected CW-complexes which have the same $(n-1)$-type form a connected tree. Actually, there is a general proof of the proper version of Whitehead's theorem for all $n \geqslant 2$ by Zobel in [28, Satz 6.11]. Notwithstanding, we include here an alternative proof of Proposition 6.3 which is purely homotopical and much closer to the material introduced in this paper.

If X^2 is a proper co-H-space, Proposition 6.3 yields a proper homotopy equivalence (under $\mathbb{R}_+$) $X^2 \vee S^2_\gamma \simeq B(\underline{L} \vee \underline{P}) \vee S^2_\delta$ (Corollary 6.4). We conjecture that in this case $S^2_\gamma = \mathbb{R}_+$ can be chosen to be the trivial spherical object. This conjecture is the proper analogue of the two-dimensional case of a conjecture due to Ganea claiming that any ordinary co-H-space has the same homotopy type as a wedge of circles and a simply connected space. Recently Iwase [22] has proved that although Ganea's conjecture is false in general, it holds for two-dimensional finite CW-complexes; that is, a finite 2-complex which is a co-H-space is homotopy equivalent to a finite wedge $\bigvee_{\alpha\in A} S^1_\alpha \vee (\bigvee_{\beta\in B} S^2_\beta)$; see [23].

2. Proper L–S category

Throughout this paper we deal with the category $\mathcal{P}$ of locally path connected, locally compact σ-compact Hausdorff spaces and proper maps. Recall that a *proper map* (*p-map*) is a continuous map $f : X \to Y$ such that $f^{-1}(K)$ is compact for each compact subset $K \subset Y$.

All maps and homotopies are assumed to be proper unless stated otherwise. We use the symbol $\simeq$ for proper homotopy and $\mathcal{P}/\simeq$ stands for the corresponding homotopy category. To ease the reading we collect in Appendix A the basic facts on proper algebraic topology used in this paper.

Given a space X in $\mathcal{P}$, a *system of ∞-neighbourhoods* of X is a decreasing sequence $\{W_j\}$ of subsets of X where the closures $K_j = \overline{X - W_j}$ form an increasing sequence of compact subsets with $K_j \subset \operatorname{int} K_{j+1}$ and $X = \bigcup \operatorname{int} K_j$.

Lemma 2.1 [6, 1.2]. *Let $A_k \subset X$ $(k \geqslant 1)$ be a locally finite sequence of compact subsets of a space X in $\mathcal{P}$. Then the sequence of compact sets $\{K_j\}$ can be chosen with the property that each A_k is contained in the interior of some difference $K_j - K_{j-1}$ $(K_0 = \emptyset)$.*

A *Freudenthal end* of a connected space X in $\mathcal{P}$ is an element of the inverse limit $\mathcal{F}(X) = \varprojlim \mathcal{U}(W_j)$, where $\mathcal{U}(-)$ stands for the family of unbounded (path) connected components. A subset $A \subset X$ is termed unbounded if the closure $\bar{A}$ is non-compact. If $\mathcal{F}(X) = \{*\}$ then X is said to be *one-ended*.

Remark 2.2. In the category $\mathcal{P}$ the constant map $X \to \{p\}$ is not a morphism unless X is compact. However, the role of the point is played in $\mathcal{P}$ by the half-line $\mathbb{R}_+ = [0, \infty)$ since there is a unique proper homotopy class of proper maps into $\mathbb{R}_+$; see [17, 6.3.5].

A proper map $\mathbb{R}_+ \to X$ is called a *ray* in X. Recall that a *properly based space* in $\mathcal{P}$ is a pair (X, α) where $\alpha : \mathbb{R}_+ \to X$ is a ray. Moreover, (X, α) is said to be *properly well based* if α is a proper cofibration. Here by a proper cofibration we mean a proper map with the proper homotopy extension property (PHEP). Let us denote cofibrations by the arrow "$\rightarrowtail$".

Remark 2.3. By use of the mapping cylinder M_α it is readily checked that any proper cofibration $\alpha : \mathbb{R}_+ \to X$ is a closed embedding and by Tietze extension arguments similar to [17, 6.3.5] it follows that α admits a proper retraction $r : X \to \mathbb{R}_+$ which is unique up to proper homotopy rel α.

A proper map $f : X \to Y$ is said to be (*properly*) *inessential* if there exists a commutative diagram in $\mathcal{P}/\simeq$ (called a *deformation diagram*)

$$\begin{tikzcd} & * \arrow[d, "\alpha"] \\ X \arrow[ur, "\beta"] \arrow[r, "f"'] & Y \end{tikzcd} \tag{1}$$

where $*$ is either $\mathbb{R}_+$ or the one-point space $\{p\}$. We also say that f can be (*properly*) *deformed to* α. Notice that in case $* = \{p\}$ then X is necessarily compact.

Given a space X in $\mathcal{P}$ a closed subset $A \subset X$ is called (*properly*) *inessential* if the inclusion $i : A \to X$ is an inessential map. A set $A \subset X$ is called (*properly*) *categorical* if $A \subset U$ with U an open set in X and the closure $\overline{U}$ is inessential. An open cover $\{U_\alpha\}$ of X is said to be (*properly*) *categorical* if each $\overline{U_\alpha}$ is an inessential set. As it is defined in [4], the *proper Lusternik–Schnirelmann category* of X, $p - \mathrm{cat}(X)$, is the least number n such that X admits a categorical open cover $\mathfrak{U} = \{U_1, U_2, \ldots, U_n\}$ with n elements. In case X is compact $p - \mathrm{cat}(X) = \mathrm{cat}(X)$ is the ordinary L–S category of X. It is not hard to check that $p - \mathrm{cat}(-)$ is a proper homotopy invariant; in fact, if $f : X \to Y$ and $g : Y \to X$ are proper maps with $gf \simeq 1_X$ one has $p - \mathrm{cat}(X) \leqslant p - \mathrm{cat}(Y)$.

Remark 2.4. There is no loss of generality in assuming that the cover $\mathfrak{U}$ above is admissible; that is, the family of components of each U_i is locally finite (and hence countable); see [6, 2.2]. Henceforth all categorical covers are assumed to be admissible. Moreover we will restrict ourselves to the class of (metrizable) ANR-spaces in $\mathcal{P}$. Recall that a metrizable space X is said to be an absolute neighbourhood retract (ANR-space) if for any metrizable space Y and any continuous map $f : A \to X$, with $A \subseteq Y$ closed, f admits a continuous extension $\tilde{f} : U \to X$ for some neighbourhood U of A. As in ordinary homotopy theory closed covers can be also used to define the proper L–S category of ANR-spaces [6, 1.5]. Furthermore, for polyhedra in $\mathcal{P}$ one can use polyhedral covers (i.e., consisting of subpolyhedra) in the definition of proper L–S category [4, 1.5].

If X is an ANR-space then any closed embedding $A \subset X$ is a proper cofibration. This is an immediate consequence of the fact that ANR-spaces in $\mathcal{P}$ have the PHEP with respect to all pairs (Y, B) where B is closed and Y is metrizable [6, 1.4]. The PHEP yields the following technical lemmas.

Lemma 2.5 [6, 2.1]. *Let X be an ANR-space in $\mathcal{P}$ and let $f : X \to Y$ be an inessential map with H: $\alpha\beta \simeq f$ as in diagram* (1) *above. Given locally finite sequences $A = \{x_j\}_{j \geqslant 1} \subset X$ and $\{t_j\}_{j \geqslant 1} \subset \mathbb{R}_+$, we can assume without loss of generality that $\alpha(t_j) = f(x_j)$, $\beta(x_j) = t_j$ and H is relative to A.*

Corollary 2.6. *Let X be a one-ended ANR-space in $\mathcal{P}$ and U an inessential closed subset in X. If each component of U is compact and the family formed by all components is locally finite in X then U can be deformed to any ray $\alpha : \mathbb{R}_+ \to X$.*

Proof. By Lemma 2.5 each component $D \subset U$ is contractible in X to a point $x_D \in D$ rel. x_D. Moreover the set $\{x_D\}$ is discrete since the deformation is proper. As X is one-ended we can find a locally finite family of arcs $\{\gamma_D\}$ joining x_D to any ray in X and the result follows. $\square$

Similarly to Lemma 2.5 we have

Lemma 2.7. *Let X be an ANR-space in $\mathcal{P}$ and let $U \subset X$ be an inessential compact subset in X. Assume in addition that in the deformation diagram*

$$\begin{tikzcd} & \mathbb{R}_+ \arrow[d, "\alpha"] \\ U \arrow[ur, "r"] \arrow[r, "k"'] & X \end{tikzcd}$$

α is a proper cofibration and $U \cap \alpha(\mathbb{R}_+) = \alpha([p, q])$. Then r can be replaced by $\tilde{r}$ with $\tilde{r} = \alpha^{-1}$ on $A = \alpha([p, q])$ and $k \simeq \alpha\tilde{r}$ rel. A. The same holds if U is closed, non-compact and $U \cap \alpha(\mathbb{R}_+) = \alpha([p, \infty))$.

Proof. As α is a proper cofibration, α is a closed embedding (Remark 2.3); moreover since all proper maps into $\mathbb{R}_+$ are properly homotopic we can assume $r(U) = [p, q]$. Here we use the Tietze extension theorem. Next we consider the proper map $G'' = r \cup G' : U \times \{0\} \cup A \times I \to \mathbb{R}_+$ where G' is any homotopy between $k|A$ and $(\alpha|A)^{-1}$. By the PHEP we extend G'' to a proper homotopy $G : U \times I \to \mathbb{R}_+$. Then $\tilde{r} = G_1$ restricts to α^{-1} on A. In order to obtain a relative homotopy we start with a homotopy H: $k \simeq \alpha\tilde{r}$ and we consider the map

$$H' = H \cup F \cup \Gamma : U \times I \times \{0\} \cup U \times \{1\} \times I \cup A \times I \times I \to X$$

where F is the homotopy $F(x, 1, t) = H(\alpha\tilde{r}(x), 1 - t)$. Notice that $\alpha\tilde{r}(x) \in U$ since $\tilde{r}(x) \in [p, q]$ if $x \in U$. Notice also that $\alpha\tilde{r}\alpha(s) = \alpha(s)$ for $p \leqslant s \leqslant q$ yields $F(\alpha(s), 1, t) = H(\alpha(s), 1 - t)$ and $F(x, 1, 0) = H(\alpha\tilde{r}(x), 1) = \alpha\tilde{r}\alpha\tilde{r}(x) = \alpha\tilde{r}(x)$. Finally Γ above is any extension of $H \cup F|A \times (I \times \{0\} \cup \{1\} \times I)$ with $\Gamma(a, 0, t) = \Gamma(a, t, 1) = a$ for each $a \in A$. By the PHEP we extend H' to a homotopy $\widehat{H}$: $U \times I \times I \to X$ such that $\widetilde{H} = \widehat{H}|X \times I \times \{1\}$: $k \simeq \alpha\tilde{r}$ rel. A. The case where U is non-compact is similar. $\square$

We devote the rest of this section to two special classes of covers which will turn out to be enough to describe the proper L–S category; namely

Definition 2.8. Let $\mathfrak{U} = \{U_k\}$ be a closed cover of a space X in $\mathcal{P}$; we say that $\mathfrak{U}$ is *a compactly patched cover* if for each k all components of U_k are compact. On the other hand, $\mathfrak{U}$ is said to be a *non-compactly patched cover* if for each k all components of U_k are non-compact.

Lemma 2.9. *Let X be a one-ended ANR-space in $\mathcal{P}$ and let $\mathfrak{U} = \{U_1, \ldots, U_n\}$ be a categorical cover for which U_1 has only compact components. Then there is a compactly patched categorical cover $\{U_1', \ldots, U_n'\}$.*

Proof. After reordering, if necessary, let $U_1, \ldots, U_k$ ($k \leqslant n$) be the elements in $\mathfrak{U}$ whose components are all compact. Since X is connected there exist $k' \geqslant k + 1$ and $j \leqslant k$ with $U_{k'} \cap U_j \neq \emptyset$. We assume $k' = k + 1$ and $j = k$ and so the intersection $U_{k+1} \cap U_k$ consists of pairwise disjoint compact sets in U_{k+1}. By Lemma 2.1 and Remark 2.4 we find an increasing sequence of compact sets $K_1 \subset K_2 \subset \cdots$ in U_{k+1} such that each component of $U_k \cap U_{k+1}$ lies in some difference $D_j = \operatorname{int} K_j - K_{j-1}$ for $j \geqslant 1$ and $K_0 = \emptyset$. Then we consider small pairwise disjoint neighbourhoods Ω_j of the frontiers $\operatorname{Fr} K_j$. Hence, the sets $U_k' = U_k \cup (\bigcup_{j=1}^{\infty} \Omega_j)$ and $U_{k+1}' = \overline{U_{k+1} - (\bigcup_{j=1}^{\infty} \Omega_j)}$ are inessential sets in X whose components are all compact. Therefore we can replace the cover $\mathfrak{U}$ by the new categorical cover $\{U_1, \ldots, U_{k-1}, U_k', U_{k+1}', U_{k+2}, \ldots, U_n\}$, and the result follows inductively. □

Lemma 2.10. *Let X be a one-ended ANR-space in $\mathcal{P}$ and let $W \subset X$ be an inessential closed subset with at least one non-compact component. Then there exists an inessential closed subset $\widetilde{W}$ containing W such that all components of $\widetilde{W}$ are non-compact.*

Proof. Let $K_1 \subset K_2 \subset \cdots$ be an increasing sequence of compact subsets with $K_j \subset \operatorname{int} K_{j+1}$ and $X = \bigcup_{j=1}^{\infty} K_j$. Let $\mathcal{B}(W)$ denote the (locally finite) family of compact components of W; see Remark 2.4. By Lemma 2.1 we can assume without loss of generality that each $L \in \mathcal{B}(W)$ is contained in some difference $\operatorname{int} K_j - K_{j-1}$ with $K_0 = \emptyset$. Then we construct a categorical set $\widetilde{W}$ with $\mathcal{B}(\widetilde{W}) = \emptyset$ and $W \subset \widetilde{W}$ as follows.

We order $\{L_1, L_2, \ldots\}$ the components in $\mathcal{B}(W)$ in such a way that the components in $\operatorname{int} K_j - K_{j-1}$ precede the components in $\operatorname{int} K_{j+1} - K_j$. Then we join L_1 to a non-compact component of W by a "K_1-controlled" arc $\gamma_1 \subset X$; that is, if γ_1 leaves K_1 then it does not return to K_1 nevermore. We consider the subarc $\gamma_1' \subset \gamma_1$ which runs from L_1 to the first non-compact component of W hit by γ_1. Moreover, the arc γ_1' is chopped to obtain small subarcs connecting the components touched by γ_1'. This way we have constructed a "string" of compact components $\mathcal{T}_1 \subset \mathcal{B}(W)$ connected by arcs (with pairwise disjoint interior) ending at some non-compact component of W.

Next, we consider the first component $L_{t_1} \in \mathcal{B}(W)$ which is not hit by the "string" $\mathcal{T}_1$. Let $K_{j_1} - K_{j_1-1}$ be the difference containing L_{t_1}. We proceed as above by constructing a new "K_{j_1}-controlled" arc γ_2 outside K_{j_1-1} from L_{t_1} to some non-compact component of W and it stops whenever it touches either a non-compact component or any point in the previous "string" $\mathcal{T}_1$. This way we can construct from γ_2, the compact components in $\mathcal{B}(W)$ hit by γ_2 and $\mathcal{T}_1$ a finite "forest" (i.e., disjoint union of "trees") whose vertices correspond to components of W and each tree contains exactly one vertex which is a non-compact component.

By proceeding in this way we obtain a locally finite family $\{T_k\}_{k \geqslant 1}$ of finite trees. Notice that the trees are finite since the arcs used in the above constructions are "controlled" by an increasing sequence of compact subsets. Moreover, each tree T_k contains a unique vertex $v_k \in T_k$ corresponding to a non-compact component. By using v_k as a root vertex we can order the vertices of T_k from v_k to the terminal vertices. Then we choose $\widetilde{W}$ to be the union of W and all edges in $\bigcup_{k=1}^{\infty} T_k$. Since each compact component $L \in \mathcal{B}(W)$ is contractible to any point $* \in L$ (rel. $*$) by Lemma 2.5, we can properly deform $\widetilde{W}$ to the union of the non-compact components of W by alternating for each T_k deformations to point of terminal vertices (i.e., compact components) and collapses of edges. □

Lemma 2.10 yields

Proposition 2.11. *Let X be a one-ended ANR-space in $\mathcal{P}$ and let $\mathfrak{U} = \{W_k\}_{k=1}^{n}$ be a categorical cover such that each W_k has at least a non-compact component. Then $\mathfrak{U}$ can replaced by a non-compactly patched categorical cover $\{\widetilde{W}_1, \ldots, \widetilde{W}_n\}$.*

As an immediate consequence of Lemma 2.9 and Proposition 2.11 we have

Proposition 2.12. *The proper L–S category of a one-ended ANR-space in $\mathcal{P}$ is attained by using only compactly and non-compactly patched categorical covers.*

3. Properly based L–S category

In the deformation diagram (1) in Section 1 the proper homotopy class of the map β is unique up to homotopy by Remark 2.2. However, for $* = \mathbb{R}_+$ the proper homotopy class of the map α in diagram (1) depends on the set of proper homotopy classes $[\mathbb{R}_+, Y]$. Each class $[\alpha] \in [\mathbb{R}_+, Y]$ is called a *strong end* of Y. When $[\mathbb{R}_+, Y]$ consists of only one element we say that Y is *strongly one-ended*. Clearly each strong end defines a Freudenthal end. More precisely there exists an onto map $[\mathbb{R}_+, Y] \to \mathcal{F}(Y)$.

In particular, the non-compact closures of the open sets in a categorical cover can be deformed to rays defining possibly different strong ends. It is obvious that for strongly one-ended spaces all rays in the deformation diagrams can be chosen to be the same. Actually the same holds for one-ended spaces; namely,

Proposition 3.1 [11, Proposition 1.6]. *Let X be a connected one-ended polyhedron* (*and more generally an ANR-space*) *in $\mathcal{P}$ with $p - \mathrm{cat}(X) = n$. Then there exists a categorical cover $\{U_1, U_2, \ldots, U_n\}$ of X such that all U_i's are deformed to the same ray α.*

The choice of a "base point" in proper homotopy theory is a subtle question because of the existence of different strong ends for the same Freudenthal end. In order to deal appropriately with this problem we introduce the notion of "pointed" proper L–S category.

Definition 3.2. Let X be a one-ended space in $\mathcal{P}$. Given a strong end $\xi \in [\mathbb{R}_+, X]$ we define the *properly based L–S category* $p - \mathrm{cat}_\xi(X)$ to be the smallest number n for which there exists a categorical cover $\{U_1, \ldots, U_n\}$ such that in the deformation diagrams

$$\begin{tikzcd} & \mathbb{R}_+ \arrow[d, "\alpha_i"] \\ U_i \arrow[ur, "r"] \arrow[r, "k_i"'] & X \end{tikzcd}$$

all rays α_i ($i \leqslant n$) define the strong end ξ.

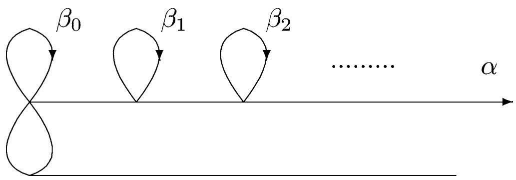
Remark 3.3. According to Proposition 3.1 we have $p - \mathrm{cat}(X) = \min\{p - \mathrm{cat}_\xi(X);\ \xi \in [\mathbb{R}_+, X]\}$ for any one-ended ANR-space X in $\mathcal{P}$.

Although for any one-ended space X in $\mathcal{P}$ the set of strong ends is in 1–1 correspondence with the (pointed) set $\varprojlim^1 \mathrm{pro} - \pi_1(X, \alpha)$ [26], the difference among the values of the properly based L–S category of X is at most one. Namely,

Proposition 3.4. *Let X be a one-ended ANR-space in $\mathcal{P}$ and let $\xi, \xi' \in [\mathbb{R}_+, X]$ be two strong ends. Then $p - \mathrm{cat}_{\xi'}(X) \leqslant p - \mathrm{cat}_\xi(X) + 1$.*

Proof. Given any categorical cover $\mathfrak{U} = \{U_1, \dots, U_n\}$ of X with $p - \mathrm{cat}_\xi(X) = n$, we consider a sequence $K_j \subset \mathrm{int}\, K_{j+1}$ of compact subsets in X with $X = \bigcup_{j=1}^\infty K_j$ and the two disjoint intersections $U_1' = U_1 \cap (\bigcup_{j=1}^\infty D_{2j-1})$ and $U_1'' = U_1 \cap (\bigcup_{j=1}^\infty D_{2j})$ where $D_j = \overline{K_j - K_{j-1}}$ with $K_0 = \emptyset$. Then $\mathfrak{U}' = \{U_1', U_1'', U_2, \dots, U_n\}$ is a new categorical cover in which U_1' (and U_1'') contains only compact components. Applying Lemma 2.9 we obtain a categorical compactly patched cover with $n + 1$ elements and, by Corollary 2.6, $p - \mathrm{cat}_{\xi'}(X) \leqslant n + 1$ for the strong end $\xi' \in [\mathbb{R}_+, X]$. $\square$

Example 3.5. A simple example of one-ended space with $p - \mathrm{cat}_\xi(X) \neq p - \mathrm{cat}(X)$ is the following two-dimensional complex $X^2 = S^1 \times \mathbb{N} \cup_{\mathbb{N}} S^1 \times \mathbb{R}_+ \subset \mathbb{R}^3$



The base ray $\alpha : \mathbb{R}_+ \to X$ is the inclusion of the axis $\{(0, 0, t);\ t \in \mathbb{R}_+\}$ in X. The base ray $\beta : \mathbb{R}_+ \to X$ however winds around all 1-spheres β_i; that is, β is given by the sum $\beta = \beta_0 + [0, 1] + \beta_1 + [1, 2] + \beta_2 + [2, 3] + \cdots$. It is immediate to check that $p - \mathrm{cat}(X) = p - \mathrm{cat}_{[\alpha]}(X) = 2$. However, $p - \mathrm{cat}_{[\beta]}(X) = 3$. Indeed, any categorical cover of X with two elements $\{P_1, P_2\}$ cannot be compactly patched since otherwise, for any ray γ, pro $-$ $\pi_1(X, \gamma)$ should be a free tower by Corollary 4.3 below, and so is its abelianized tower pro $- \pi_1(X, \gamma)^{ab} =$ pro $- H_1(X)$, but $\varprojlim \mathrm{pro} - H_1(X) \cong \mathbb{Z}$. Hence P_1 or P_2 contains an embedded ray $\mathbb{R}_+ \subset X$ which necessarily lies in the cylinder $S^1 \times \mathbb{R}_+$ and so represents the strong end $[\alpha] \neq [\beta]$. Therefore, $p - \mathrm{cat}_{[\beta]}(X) = 3$ by Proposition 3.4.

The following domination property holds for the properly based L–S category.

Lemma 3.6. *Let (X, α) and (Y, β) be properly based spaces in $\mathcal{P}$ for which there is a proper map $f : X \to Y$ with $gf \simeq \mathrm{id}_X$ and $f\alpha \simeq \beta$ (and hence $g\beta \simeq \alpha$). Then $p - \mathrm{cat}_{[\alpha]}(X) \leqslant p - \mathrm{cat}_{[\beta]}(Y)$.*

Proof. Let $\{U_1, \dots, U_n\}$ be a categorical cover of Y for which $p - \mathrm{cat}_{[\beta]} = n$. We claim that each $f^{-1}(U_i)$ is properly contractible in X to α. Indeed, if G: $gf \simeq \mathrm{id}_X$ and H^i: $k_{\overline{U}_i} \simeq \beta r$ is a proper deformation for $\overline{U}_i$ then $\widetilde{H}^i = gH^i(f \times \mathrm{id}_I)$: $k_{f^{-1}(U_i)} \simeq \alpha r f$. In addition, if $F\colon\ g\beta \simeq \alpha$ we obtain by track addition a deformation $F\#\widetilde{H}^i : f^{-1}(\overline{U}_i) \times I \to X$ which carries $f^{-1}(\overline{U}_i)$ to the ray α. $\square$

As a consequence we have

Lemma 3.7. *Let (X,α) be any properly based space in $\mathcal{P}$. Then there exists a homotopy equivalence $k\colon\ X \simeq \widetilde{X}$ such that $k\alpha$ is properly homotopic to a cofibration $\tilde{\alpha} : \mathbb{R}_+ \rightarrowtail \widetilde{X}$ and $p - \mathrm{cat}_{[\alpha]}(X) = p - \mathrm{cat}_{[\tilde{\alpha}]}(\widetilde{X})$.*

Proof. It is enough to consider the mapping cone $\widetilde{X} = M_\alpha$ of α and $\tilde{\alpha} : \mathbb{R}_+ \rightarrowtail \widetilde{X}$ the inclusion $x \mapsto (x,0) \in M_\alpha$. $\square$

Next proposition provides us with properly well based categorical covers

Proposition 3.8. *Given a properly based one-ended ANR-space (X,β), let $\mathfrak{U} = \{U_1, \dots, U_n\}$ be a categorical cover of X properly deformable to β. Then there exist a proper cofibration $\alpha : \mathbb{R}_+ \rightarrowtail X$ and a non-compactly patched categorical cover $\mathfrak{U}' = \{\Omega_1, \dots, \Omega_n\}$ of properly well based closed subsets (Ω_i, α). Moreover, if $\mathfrak{U}$ is not compactly patched then $\alpha \simeq \beta$ represents the same strong end as β. If on the contrary, $\mathfrak{U}$ is compactly patched then α may not represent the same strong end as β but each intersection $\Omega_i \cap \Omega_j$ has exactly one non-compact component.*

Proof. We consider two cases.

(a) Suppose $\mathfrak{U}$ is not compactly patched, and let C be a non-compact component of, say, U_1. If $r : X \to \mathbb{R}_+$ is any proper map and $\alpha : \mathbb{R}_+ \rightarrowtail C$ is any proper cofibration, it follows that $\beta \simeq \alpha$ since $r\alpha \simeq \mathrm{id}_{\mathbb{R}_+}$, and hence α represents the same strong end as β. Let $W_1 = U_1$. Assume inductively that we have constructed closed subsets $W_1, \dots, W_k$ with $\bigcup_{i=1}^k W_i = \bigcup_{k=1}^k U_i$, $\alpha(\mathbb{R}_+) \subset W_1 \cap \dots \cap W_k$ and each W_i is properly deformable to α. Then we consider the disjoint union $W_{k+1} = U'_{k+1} \cup \alpha(\mathbb{R}_+)$ where $U'_{k+1} = \overline{U_{k+1} - \bigcup_{i=1}^k W_i}$. Notice that U'_{k+1} is properly deformable to β and hence to α. Therefore W_{k+1} is also properly deformable to α and the result follows. In order to obtain from $\{W_i\}_{1 \leqslant i \leqslant n}$ the non-compactly patched cover $\mathfrak{U}'$ we proceed as in the proof of Lemma 2.10.

(b) Assume the cover $\mathfrak{U}$ is compactly patched. By Corollary 2.6 we can assume β to be a cofibration (i.e., a closed embedding $\mathbb{R}_+ \subset X$). If $D_1, \dots, D_n, \dots$ are the components of U_1 we denote by $x_1^\beta = \beta(s_1^\beta)$ and $y_1^\beta = \beta(t_1^\beta)$ the first and the last points in $\beta(\mathbb{R}_+) \cap D_1$ in case this intersection is not empty. We replace β by a new cofibration β_1 defined by replacing the arc $\beta([s_1^\beta, t_1^\beta])$ by a new arc $\gamma_1 : [s_1^\beta, t_1^\beta] \to D_1$ in D_1. As all components D_j are compact we have a deformation diagram

$$\begin{tikzcd} & \mathbb{R}_+ \arrow[d, "\beta_1"] \\ U_1 \arrow[ur, "r_1"] \arrow[dr, "i_1"'] & \\ & X \end{tikzcd}$$

Moreover, we can assume that $r_1 = \beta_1^{-1}$ on $\beta_1([s_1^\beta, t_1^\beta])$ and the proper homotopy $H^1\colon\ i_1 \simeq \beta_1 r_1$ is relative the arc γ_1. For this we use Corollary 2.6 and Lemma 2.7.

Hence, we can extend this deformation to the union $U_1 \cup \beta_1(\mathbb{R}_+)$. Next, we consider $x_2^{\beta_1} = \beta_1(s_2^{\beta_1})$ and $y_2^{\beta_1} = \beta_1(t_2^{\beta_1})$ the first and the last points in $\beta_1(\mathbb{R}_+) \cap D_2$. Notice that $[s_2^{\beta_1}, t_2^{\beta_1}] \cap [s_1^{\beta_1}, t_1^{\beta_1}] = \emptyset$ and we can replace β_1 by a new cofibration β_2 which agrees with β_1 outside D_2 and inside D_2 is an arc γ_2 running from $x_2^{\beta_1}$ to $y_2^{\beta_1}$. We can also obtain a proper homotopy H^2: $i_1 \simeq \beta_2 r_2$ relative to the arc γ_2, where $r_2 = \beta_2^{-1}$ on $\beta_2(\mathbb{R}_+) \cap (D_1 \cup D_2)$, and $H^2 = H^1$ on D_1.

Proceeding inductively in this way we get a sequence of arcs $\gamma_i \subset D_i$, cofibrations $\beta_i : \mathbb{R}_+ \rightarrowtail X$ with $\beta_i|[s_i^{\beta_{i-1}}, t_i^{\beta_{i-1}}] = \gamma_i$ and $\beta_i = \beta_{i-1}$ outside D_i, proper maps $r_i : U_1 \to \mathbb{R}_+$ with $r_i = \beta_i^{-1}$ on $\beta_i(\mathbb{R}_+) \cap (D_1 \cup D_2 \cup \cdots \cup D_i)$, and proper homotopies H^i: $i_1 \simeq \beta_i r_i$ rel. γ_i extending H^{i-1}. From these data we define a cofibration $\alpha : \mathbb{R}_+ \rightarrowtail X$ with $\alpha = \beta_i$ on $[s_i^{\beta_{i-1}}, t_i^{\beta_{i-1}}]$ and $\alpha = \beta$ outside $Z = \bigcup_{i=1}^{\infty} [s_i^{\beta_{i-1}}, t_i^{\beta_{i-1}}]$ (we set $\beta_0 = \beta$), a proper map $r : U_1 \to \mathbb{R}_+$ with $r = \alpha^{-1}$ on $\alpha(\mathbb{R}_+) \cap U_1$, and a proper homotopy H: $i_1 \simeq \alpha r$ relative $\alpha(\mathbb{R}_+) \cap U_1$. Hence, we have shown that the union $\widetilde{U}_1 = U_1 \cup \alpha(\mathbb{R}_+)$ is properly deformable to α. The cover $\{\widetilde{U}_1, U_2, \ldots, U_n\}$ is then in case (a) and from this cover we get the cover $\{W_i\}_{1 \leqslant i \leqslant n}$ above. Notice that, in this case, each intersection $W_i \cap W_j$ has only one non-compact component which is the one containing $\alpha(\mathbb{R}_+)$. Now the proof of Lemma 2.10 shows how to obtain the non-compactly patched cover $\mathfrak{U}' = \{\Omega_i\}_{1 \leqslant i \leqslant n}$ and that each component of $\Omega_i \cap \Omega_j$ has exactly one non-compact component. $\square$

For the next result we need some extra notation. Recall that the *proper wedge* $X \vee_{(\alpha,\beta)} Y$ of two properly based spaces (X, α) and (Y, β) where α or β is a proper cofibration is the push-out of the diagram $X \xleftarrow{\alpha} \mathbb{R}_+ \xrightarrow{\beta} Y$. If $\alpha = \beta$ we write $X \vee_\alpha Y$ and we simply write $X \vee Y$ if the base ray is clear from the context.

Proposition 3.9. *Let (X, β) be a properly based one-ended ANR-space in $\mathcal{P}$ with $p - \mathrm{cat}_{[\beta]}(X) = m$. Then there exist a proper cofibration $\alpha : \mathbb{R}_+ \rightarrowtail X$ and a proper homotopy equivalence under $\mathbb{R}_+$*

$$Y \simeq X \vee_\alpha \Sigma_{\mathbb{R}_+} P_1 \vee_\alpha \Sigma_{\mathbb{R}_+} P_2 \vee_\alpha \cdots \vee_\alpha \Sigma_{\mathbb{R}_+} P_m$$

where (P_j, α) are properly well based closed subsets of X and (Y, α) is a properly well based ANR-space which can be covered by m closed subsets $\{Y_j\}_{1 \leqslant j \leqslant m}$ with $Y_j \simeq \mathbb{R}_+$. In addition, if the categorical cover for which $p - \mathrm{cat}_{[\beta]}(X) = m$ is not compactly patched then the cofibration α can be chosen to represent the same strong end as the original base ray β.

Proof. By Proposition 3.8 we have $X = \bigcup_{j=1}^{m} P_j$ where each P_j is properly deformable to a cofibration $\alpha : \mathbb{R}_+ \rightarrowtail X$ with $\alpha(\mathbb{R}_+) \subset P_j$ for all j. Moreover, by Lemma 2.7 the deformation can be carried out relative $\alpha(\mathbb{R}_+)$. Then we set $Y = X \cup C_{\mathbb{R}_+} P_1 \cup C_{\mathbb{R}_+} P_2 \cup \cdots \cup C_{\mathbb{R}_+} P_m$. By using the gluing Lemma A.1 we get the required proper homotopy equivalence. $\square$

Remark 3.10. Proposition 3.9 can be used to give a relationship between proper L–S category and proper strong L–S category. Recall that the proper strong L–S category of X

is the smallest number $k = p - \mathrm{Cat}(X)$ such that the proper homotopy type of X can be represented by a polyhedron Y which can be covered with k subpolyhedra $Y_i \simeq \mathbb{R}_+$. Indeed, let β be a ray with $p - \mathrm{cat}(X) = p - \mathrm{cat}_{[\beta]}(X)$, see Remark 3.3. Then we attach the proper (based) cone over $Z = \Sigma_{\mathbb{R}_+} P_1 \vee_\alpha \cdots \vee_\alpha \Sigma_{\mathbb{R}_+} P_m$ to Y via the proper homotopy equivalence in Proposition 3.9. The gluing Lemma A.1 yields a homotopy equivalence $X \simeq Y' = Y \cup C_{\mathbb{R}_+} Z$ where Y' has proper strong L–S category $\leqslant m + 1$. Hence for any one-ended ANR-space in $\mathcal{P}$ we get $p - \mathrm{Cat}(X) \leqslant p - \mathrm{cat}(X) + 1$. This extends Proposition 2.9 in [6] to one-ended spaces.

4. Spaces with proper L–S category $\leqslant 2$

It is a well-known result in ordinary homotopy that the class of well pointed spaces with L–S category $\leqslant 2$ coincides with the class of co-H-spaces. From this one can derive that the fundamental group of a space X with $\mathrm{cat}(X) \leqslant 2$ is always free; see [24].

We start with the following

Proposition 4.1. *Let (X, α) be a properly well based ANR-space in $\mathcal{P}$. Then $p - \mathrm{cat}_{[\alpha]}(X) \leqslant 2$ if and only if (X, α) is a proper co-H-space.*

Here by a *proper co-H-space* we mean a properly well based space (X, α) for which there exist a proper map $\mu : X \to X \vee_\alpha X$ and proper homotopies $(1_X, \alpha r)\mu \simeq \mathrm{id}_X$ and $(\alpha r, 1_X)\mu \simeq \mathrm{id}_X$. Here r is any retract of α and (f, g) denotes the map defined by applying the push-out property to f and g.

Proof of Proposition 4.1. Let $r : X \to \mathbb{R}_+$ be a proper retraction of α and assume $p - \mathrm{cat}_{[\alpha]}(X) \leqslant 2$; that is, X is the union of two inessential closed subsets $X = U_1 \cup U_2$. Let us consider deformation diagrams ($i = 1, 2$)

$$\begin{tikzcd} & \mathbb{R}_+ \arrow[dd, "\alpha"] \\ U_i \arrow[ur, "r|U_i"] \arrow[dr, "k_i"'] & \\ & X \end{tikzcd}$$

with homotopies F^i: $k_i \simeq \alpha r$. By using the PHEP we can extend F^i to a proper homotopy $H^i : X \times I \to X$ with $H^i_0 = \mathrm{id}_X$. Let $\mu_2 : U_2 \to X$ and $\mu_1 : U_1 \to X$ be the corresponding restrictions of H^1_1 and H^2_1 respectively. Notice that both μ_2 and μ_1 agree with αr on $U_1 \cap U_2$ and hence we have a well defined proper map $\mu : X \to X \vee_\alpha X$ given by $\mu|U_1 = \mu_1$ and $\mu|U_2 = \mu_2$. We claim that (X, μ) is a proper co-H-space. Indeed, the composite $(\mathrm{id}_X, \alpha r)\mu$ agrees with μ_1 on U_1 and with $\alpha r \mu_2$ on U_2. The latter is properly homotopic to $\alpha r|U_2$ via $\alpha r H^1|U_2 \times I$. Moreover, $\alpha r H^1$ restricts to a homotopy $\alpha r H^{12}$: $\alpha r \simeq \alpha r \alpha r = \alpha r$ on $U_1 \cap U_2$. We now consider any proper extension $\widetilde{H}^{12} : (U_1 \cap U_2) \times I \times I \to \mathbb{R}_+$ of

$$r H^{12} \cup r \times \big(\{0,1\} \times \mathrm{id}_I \cup \mathrm{id}_I \times \{1\}\big) : U_1 \cap U_2 \times \big(I \times \{0\} \cup \{0,1\} \times I \cup I \times \{1\}\big) \\ \to \mathbb{R}_+.$$

Here we use that $\mathbb{R}_+$ is properly "contractible"; see Remark 2.2. Now by the PHEP we can extend

$$\alpha r H^1 \cup \alpha r \tilde{\mu}_2 \cup \alpha \widetilde{H}^{12} : U_2 \times \big(I \times \{0\} \cup \{0,1\} \times I\big) \cup (U_1 \cap U_2) \times I \times I \to X,$$

with $\tilde{\mu}_2(x, \varepsilon, t) = \mu_2(x)$ $(\varepsilon = 0, 1)$, to a proper map $\widetilde{H}^1 : U_2 \times I \times I \to X$ such that the restriction $\widehat{H}^1_1 : U_2 \times I \times \{1\} \to X$ is a homotopy $\widehat{H}^1_1$: $\alpha r \mu_2 \simeq \alpha r|U_2$ which is the constant homotopy αr on $U_1 \cap U_2$. Moreover $\widehat{H}^1_1$ extends to a homotopy G: $(\mathrm{id}_X, \alpha r)\mu \simeq \mu_1$ by setting $G|U_1 \times I = \mu_1 \times \mathrm{id}_I$. By track addition we get a proper homotopy $G\#H^2$: $(\mathrm{id}_X, \alpha r)\mu \simeq \mathrm{id}_X$. A proper homotopy $(\alpha r, \mathrm{id}_X)\mu \simeq \mathrm{id}_X$ is obtained similarly.

For the converse, let $\mu : X \to X \vee_\alpha X$ be a proper comultiplication with homotopies H^1: $\mathrm{id}_X \simeq (\mathrm{id}_X, \alpha r)\mu$ and H^2: $\mathrm{id}_X \simeq (\alpha r, \mathrm{id}_X)\mu$. Then if $\{U_1, U_2\}$ is the closed cover with $U_k = \mu^{-1}(i_k(X))$ where $i_k : X \to X \vee_\alpha X$ are the obvious inclusions $(k = 1, 2)$, we get that $H^2|U_1 \times I$ and $H^1|U_2 \times I$ are deformations to the ray α. Hence $p-\mathrm{cat}_{[\alpha]}(X) \leqslant 2$. □

We now proceed to characterize the fundamental pro-group of spaces with proper L–S category $\leqslant 2$. We start with the following proposition and its corollary; compare Proposition 4.8 in [13].

Proposition 4.2. *Let X be a one-ended polyhedron in $\mathcal{P}$ and let $X = U \cup V$ be a compactly patched categorical polyhedral cover of X. Then there exists a graph Y in $\mathcal{P}$ and proper maps $\varphi : Y \to X$ and $\psi : X \to Y$ such that $\psi\varphi \simeq \mathrm{id}_Y$ and $\varphi\psi \simeq j$: $X^1 \subset X$.*

Proof. Let $\{U_n\}_{n\geqslant 1}$ and $\{V_m\}_{m\geqslant 1}$ be the components of U and V. We choose a triangulation of X such that U and V are subcomplexes and the intersections $U_n \cap V_m$ are full subcomplexes. Next we pick vertices $u_n \in U_n - V$, $v_m \in V_m - U$ and $c_{mn} \in C_{mn}$, where C_{mn} ranges over the family of components of $U_n \cap V_m \neq \emptyset$. Then the graph Y consists of by vertices $\tilde{u}_n$, $\tilde{v}_m$ and $\tilde{c}_{mn}$ which define a 1–1 correspondence, φ_0, with the vertices in X previously chosen. Edges in Y are exactly of the form $\sigma_{mn} = (\tilde{u}_n, \tilde{c}_{mn})$ and $\gamma_{mn} = (\tilde{v}_m, \tilde{c}_{mn})$. Then the map $\varphi : Y \to X$ extends the 1–1 correspondence φ_0 by carrying the edge σ_{mn} to a simplicial path in U_n from u_n to c_{mn}, and γ_{mn} to a simplicial path in V_m from v_m to c_{mn}. The map $\psi : X^1 \to Y$ is the lineal extension of the map $\psi_0 : X^0 \to Y$ which carries vertices in $U_n - V$ to $\tilde{u}_n$, vertices in $V_m - U$ to $\tilde{v}_m$, and vertices in C_{mn} to c_{mn}.

Next we show that $\varphi\psi \simeq j$: $X^1 \subset X$ (properly). For this we define for each vertex $x \in X_x$ $(= U_n, V_m$, or $C_{mn})$ a path $H^1_x : I \to X_x$ from x to $\varphi\psi(x)$ $(= \tilde{u}_n, \tilde{v}_m$, or $\tilde{c}_{mn}$ accordingly). The map $H^1 = \bigcup_{x\in X^0} H^1_x : Y^0 \times I \to X$ is proper since the family $\{H^1_x(I)\}$ is locally finite. Given an edge $A = (x, x') \subset X$ we consider $\widetilde{H}^2_A = k_A \cup j_A \cup H^1 : S^1_A = A \times \{0, 1\} \cup \{x, x'\} \times I \to X$ where j_A: $A \subset X$ and $k_A = \varphi\psi|A$. As $\{\widetilde{H}^2_A\}$ is a locally finite family of loops in X the map $\widetilde{H}^2 = \bigcup_A \widetilde{H}^2_A$ is proper. In addition, if $\mathcal{A} = \{A;\ \widetilde{H}^2_A(S^1_A) \subset U\}$ and $\mathcal{A}' = \{A;\ \widetilde{H}^2_A(S^1_A) \subset V\} - \mathcal{A}$, we apply the fact that $\{U, V\}$ is categorical to get proper extensions H^2_U and H^2_V of $\bigcup_{A\in\mathcal{A}} \widetilde{H}^2_A$ and $\bigcup_{A\in\mathcal{A}'} \widetilde{H}^2_A$ respectively. Hence $H^2 = H^2_U \cup H^2_V : Y \times I \to X$ is a proper homotopy $\varphi\psi \simeq j$.

Finally, to show $\psi\varphi \simeq \mathrm{id}_Y$ we simply observe that $\psi\varphi = \mathrm{id}$ on vertices and for each edge σ_{mn} (γ_{mn}, respectively) we have $\psi\varphi|\sigma_{mn} \simeq \mathrm{id}_{\sigma_{mn}}$ ($\psi\varphi|\gamma_{mn} \simeq \mathrm{id}_{\gamma_{mn}}$, respectively) by a homotopy inside σ_{mn} (γ_{mn}, respectively) relative to the vertices. □

As a corollary we get

Corollary 4.3. *Let X be a one-ended polyhedron in $\mathcal{P}$ and let $X = U \cup V$ be a compactly patched categorical polyhedral cover of X. Then the fundamental pro-group* $\text{pro} - \pi_1(X, \alpha) \cong \underline{L}$ *is pro-isomorphic to a free tower.*

Proof. The maps φ and ψ in Proposition 4.2 and the proper cellular approximation theorem yield that Y is one-ended and $\psi_* : \text{pro} - \pi_1(X, \alpha) \to \text{pro} - \pi_1(Y, \psi\alpha)$. is a pro-isomorphism. Moreover, the classification of proper homotopy types of graphs in [5] implies that Y is proper homotopically equivalent to a 1-dimensional spherical object S^1_ε, for which $\text{pro} - \pi_1(S^1_\varepsilon, \eta)$ is a free tower for the natural base ray η: $\mathbb{R}_+ \subset S^1_\varepsilon$. Finally the result is consequence of the following lemma. $\square$

Lemma 4.4. *Let (X, α) be a properly base one-ended space in $\mathcal{P}$ with* $\text{pro} - \pi_1(X, \alpha)$ *a free tower. Then for any ray $\beta : \mathbb{R}_+ \to X$ the tower* $\text{pro} - \pi_1(X, \beta)$ *is also a free tower.*

Proof. By the PHEP we can assume that $\alpha(n) = \beta(n)$ for all $n \in \mathbb{N}$; compare with Lemma 2.5. By choosing a suitable system of ∞-neighbourhoods we can write $\text{pro} - \pi_1(X, \alpha) = \{\pi_1(U_i, x_i); f_i\}$ and $\text{pro} - \pi_1(X, \beta) = \{\pi_1(U_i, x_i); g_i\}$ with $U_0 = X$ and $x_j = \alpha(n_j) = \beta(n_j)$, $0 = n_0 < n_1 < n_2 < \cdots$. Moreover, if $G_i = \pi_1(U_i, x_i)$ the bonding homomorphisms $f_i, g_i : G_{i+1} \to G_i$ are related by the equations $g_i(x) = \rho_i f_i(x) \rho_i^{-1}$ where $\rho_i \in G_i$ is the element represented by the loop $\beta|[n_i, n_{i+1}] * \bar{\alpha}|[n_i, n_{i+1}]$. Here $\bar{\alpha}$ denotes the inverse path. After reindexing, if necessary, a pro-isomorphism $\underline{\varphi}$: $\underline{L} \cong \text{pro} - \pi_1(X, \alpha)$ yields commutative diagrams of groups; see [25, II.2.2.5]

$$\begin{tikzcd}
G_i \arrow[r] \arrow[d, "\varphi_i"'] & G_{i-1} \arrow[d, "\varphi_{i-1}"] \\
*_{j \geqslant k_i} \mathbb{Z}\langle b_j \rangle \arrow[r] & *_{j \geqslant k_{i-1}} \mathbb{Z}\langle b_j \rangle \arrow[ul, dotted, "\psi_i"']
\end{tikzcd}$$

The homomorphisms $\tilde{\psi}_i(z) = \rho_{i-1} \psi_i(z) \rho_{i-1}^{-1}$ and $\tilde{\varphi}_i = \varphi_i$ define a pro-isomorphism $\text{pro} - \pi_1(X, \beta) \cong \underline{L}'$ where

$$\underline{L}' = \left\{ *_{i=0}^{\infty} \mathbb{Z}\langle b_i \rangle \xleftarrow{j_0} *_{i=k_1}^{\infty} \mathbb{Z}\langle b_i \rangle \xleftarrow{j_1} \cdots \right\}$$

is the tower with bonding homomorphisms $j_s(b_i) = y_{i-1} b_i y_{i-1}^{-1}$ with $y_i = \varphi_i(\rho_i)$ for $i \geqslant k_{s+1}$. One readily checks that $\underline{L}'$ is pro-isomorphic to the free tower

$$\underline{L}'' = \left\{ *_{i=0}^{\infty} \mathbb{Z}\langle c_i \rangle \leftarrow *_{i=k_1}^{\infty} \mathbb{Z}\langle c_i \rangle \leftarrow \cdots \right\}$$

by the levelwise pro-isomorphism carrying c_i in the k_j-level ($k_j \leqslant i$) to $y_j .. y_i b_i (y_j .. y_i)^{-1}$. $\square$

Next proposition gives us the proper fundamental pro-group of polyhedra in $\mathcal{P}$ with proper L–S category $\leqslant 2$ (i.e., proper co-H-spaces). For this we will consider, together with free towers, another special type of towers of free groups. Namely, by a *telescopic tower* we mean a tower pro-isomorphic to a tower of the form

$$\underline{P} = \left\{ P_0 \xleftarrow{p_1} P_1 \xleftarrow{p_2} \cdots \right\}$$

where $P_i = \langle D_i \rangle$ are free groups of countable basis D_i such that $D_{i-1} \subset D_i$, and the differences $D_i - D_{i-1}$ are finite (possibly empty); moreover, the bonding homomorphisms p_k are the obvious projections.

Lemma 4.5. *Let $\underline{f} : \underline{P} \twoheadrightarrow \underline{Q}$ be an epimorphism in $(\mathcal{G}r, \mathrm{tow} - \mathcal{G}r)$ where $\underline{P}$ is a telescopic tower and $\underline{Q}$ is a tower of free groups. Then $\underline{Q}$ is a telescopic tower.*

Proof. We start by choosing a levelwise representative of $\underline{f}$

$$\begin{tikzcd}
\cdots \arrow[r] & P_{k_2} \arrow[r, "p_2"] \arrow[d, "f_2"] & P_{k_1} \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0 \arrow[d, "f_0"] \\
\cdots \arrow[r] & Q_2 \arrow[r, "q_2"] & Q_1 \arrow[r, "q_1"] & Q_0
\end{tikzcd}$$

Without loss of generality we can assume that the vertical homomorphisms are onto by replacing, if necessary, $\underline{Q}$ by the pro-isomorphic tower $\mathrm{Im}\,\underline{f}$. Therefore, the bonding homomorphisms $q_i : Q_i \to Q_{i-1}$ can be supposed to be onto. From the exact sequences $\mathrm{Ker}\, f_i \hookrightarrow P_{k_i} \to Q_i$ one readily checks that f_i induces epimorphisms $\tilde{f}_i : \mathrm{Ker}\, p_i \twoheadrightarrow \mathrm{Ker}\, q_i$. In addition, since $\mathrm{Ker}\, p_i$ is the normalized subgroup of a finitely generated free group it follows that $\mathrm{Ker}\, q_i = N(L_i)$ is also the normalized subgroup of a finitely generated free group L_i. Moreover, since $q_i : Q_i \twoheadrightarrow Q_{i-1}$ is an epimorphism between free groups, it follows from [19] that Q_i is a free product $Q_i = Q_i^1 * Q_i^2$ where Q_i^1 is mapped onto Q_{i-1} isomorphically, and Q_i^2 is carried to the trivial element. Hence, the homomorphism q_i can be identified with the first projection $Q_i^1 * Q_i^2 \twoheadrightarrow Q_i^1$.

It remains to show that Q_i^2 is finitely generated. For this we observe that the normalizer of Q_i^2, $N(Q_i^2)$, coincides with $N(L_i)$ where L_i is finitely generated, and that, if B and C are basis for L_i and Q_i^2 respectively, $N(L_i)$ and $N(Q_i^2)$ are free groups of basis $\{qbq^{-1};\ q \in Q_i^1,\ b \in B\}$ and $\{qcq^{-1};\ q \in Q_i^1,\ c \in C\}$ respectively. □

Proposition 4.6. *Let (X, α) be a properly based polyhedron in $\mathcal{P}$ with $p - \mathrm{cat}_{[\alpha]}(X) \leqslant 2$. Then $\mathrm{pro} - \pi_1(X, \alpha)$ is pro-isomorphic to a coproduct $\underline{L} \vee \underline{P}$ in $(\mathcal{G}r, \mathrm{tow} - \mathcal{G}r)$ of a free tower $\underline{L}$ with a telescopic tower $\underline{P}$.*

In the proof of Proposition 4.6 we will use the following results from [6].

Lemma 4.7 [6, 3.2]. *Let $(X, \alpha) = (\Sigma_{\mathbb{R}_+} Y; \alpha)$ be the proper suspension of a properly well based space in $\mathcal{P}$. Then $\mathrm{pro} - \pi_1(X, \alpha)$ is pro-isomorphic to a coproduct $\underline{L} \vee \underline{P}$ of a free* *tower* $\underline{L}$ *with a telescopic tower* $\underline{P}$. *Moreover, if* Y *has only non-compact components then* $\text{pro}-\pi_1(X,\alpha)\cong\underline{P}$. *If* Y *has only one non-compact component then* $\text{pro}-\pi_1(X,\alpha)\cong\underline{L}$ *is a free tower.*

Lemma 4.8 [6, 2.11]. *Let* Y *be a space in* $\mathcal{P}$ *with* $Y=Y_1\cup Y_2$ *where* Y_i *has the same proper homotopy type as* $\mathbb{R}_+$ $(i=1,2)$. *Then* $Y\simeq\Sigma(Y_1\cap Y_2)$. *Moreover, if* $Y_1\cap Y_2$ *is properly well based by* $\alpha:\mathbb{R}_+\rightarrowtail Y_1\cap Y_2$ *then* $Y\simeq\Sigma_{\mathbb{R}_+}(Y_1\cap Y_2)$ *under* $\mathbb{R}_+$.

Proof of Proposition 4.6. Consider a polyhedral categorical cover $\mathfrak{U}=\{U_1,U_2\}$ with each P_i properly deformable to the ray α. We will consider two cases:

(a) $\mathfrak{U}$ is a compactly patched cover. Then the result follows from Corollary 4.3.

(b) $\mathfrak{U}$ is not a compactly patched cover. Then by Proposition 3.8, we can assume without loss of generality that this cover is non-compactly patched. Moreover, by the proof of Proposition 3.9 there exists a proper cofibration $\beta:\mathbb{R}_+\rightarrowtail X$ with $\beta\simeq\alpha$ and a proper homotopy equivalence under $\mathbb{R}_+$

$$Y\simeq X\vee_\beta\Sigma_{\mathbb{R}_+}U_1\vee_\beta\Sigma_{\mathbb{R}_+}U_2$$

where (U_i,β) are properly well based and $Y=Y_1\cup Y_2$ with $Y_i\simeq\mathbb{R}_+$. In particular, $Y\simeq\Sigma_{\mathbb{R}_+}(Y_1\cap Y_2)$ by Lemma 4.8. By applying the Seifert–Van Kampen theorem levelwise we obtain a pro-isomorphism

$$\text{pro}-\pi_1(Y,\beta)\cong\text{pro}-\pi_1(X,\beta)\vee\text{pro}-\pi_1(\Sigma_{\mathbb{R}_+}U_1\vee_\beta\Sigma_{\mathbb{R}_+}U_2,\beta).$$

Furthermore, by Lemma 4.7 $\text{pro}-\pi_1(Y,\beta)\cong\underline{L}\vee\underline{P}$ and $\text{pro}-\pi_1(\Sigma_{\mathbb{R}_+}U_1\vee_\beta\Sigma_{\mathbb{R}_+}U_2,\beta)\cong\underline{P}'$, where $\underline{P}$ and $\underline{P}'$ are telescopic towers and $\underline{L}$ is a free tower. Thus, we get a pro-isomorphism $\underline{\varphi}:\underline{G}\vee\underline{P}'\to\underline{L}\vee\underline{P}$ where $\underline{G}=\text{pro}-\pi_1(X,\alpha)\cong\text{pro}-\pi_1(X,\beta)$. Here we use that both α and β represent the same strong end. Since $\varprojlim\underline{L}=0$ the restriction $\underline{\varphi}':\underline{P}'\to\underline{L}\vee\underline{P}$ factorizes through $\underline{P}$; that is, we have a commutative diagram in $(\mathcal{G}r,\text{tow}-\mathcal{G}r)$

$$\begin{array}{ccc}\underline{P}' & \xrightarrow{\underline{\varphi}'} & \underline{P}\\ {\scriptstyle \underline{i}_2}\downarrow & & \downarrow{\scriptstyle \underline{i}_2}\\ \underline{G}\vee\underline{P}' & \xrightarrow{\underline{\varphi}} & \underline{L}\vee\underline{P}\end{array}$$

In particular, $\underline{\varphi}'$ is a monomorphism. Thus $\underline{P}'\cong\underline{\varphi}'(\underline{P}')$ and we get a pro-isomorphism

$$\underline{G}\cong\underline{G}\vee\underline{P}'/N(\underline{P}')\cong\underline{L}\vee\underline{P}/N\big(\underline{\varphi}'(\underline{P}')\big)\cong L\vee\big(P/N_{\underline{P}}\big(\underline{\varphi}'(\underline{P}')\big)\big).$$

Here $N(-)$ denotes the corresponding tower of normalized subgroups and the pro-isomorphism on the right-hand side is induced by the epimorphism $(\text{id},\underline{q}):\underline{L}\vee\underline{P}\to L\vee(P/N_{\underline{P}}(\underline{\varphi}'(\underline{P}')))$ where $\underline{q}$ is the obvious projection whose kernel is readily checked to be the tower $N(\underline{\varphi}'(\underline{P}'))$.

As a subtower of $\underline{L}\vee\underline{P}$, $\underline{G}$ and hence the quotient tower $\underline{P}''=P/N_{\underline{P}}(\underline{\varphi}'(\underline{P}'))$ are pro-isomorphic to towers of free groups. Hence, $\underline{P}''$ is a telescopic tower by Lemma 4.5. Since $\underline{G}\cong\underline{L}\vee\underline{P}''$, Proposition 4.6 holds in case (b). $\square$

5. Proper m-dimensional L–S category

In this section we consider a weaker notion of L–S category which will allows us to prove two results: a converse to Proposition 4.6 for two-dimensional CW-complexes in $\mathcal{P}$ and the closure under retractions of the class of towers of the form $\underline{L} \vee \underline{P}$.

We start by recalling some basic fact about CW-complexes in $\mathcal{P}$. It is well known that for any finite dimensional CW-complex X in $\mathcal{P}$ there exists a (maximal) tree $T \subset X$ such that $\mathcal{F}(T) = \mathcal{F}(X^1) = \mathcal{F}(X)$. The tree T is called an *end-faithful tree*; from this result one readily derives that any 1-dimensional CW-complex in $\mathcal{P}$ has the same proper homotopy type (relative T) of a 1-spherical object S^1_α under T; see [5]. Moreover, by using based attaching maps, any finite dimensional CW-complex X in $\mathcal{P}$ with $\mathcal{F}(X) = \mathcal{F}(T)$ has the proper homotopy type (under T) of a *normalized* CW-complex Y; that is, $Y^1 = S^1_{\alpha_1}$ is a 1-spherical object and the n-skeleton Y^n is obtained as a mapping cone (under T) of a proper map $f_n : S^{n-1}_{\alpha_n} \to Y^{n-1}$. Finally, recall that any finite dimensional CW-complex X in $\mathcal{P}$ has the same proper homotopy type of a polyhedron P of the same dimension.

Definition 5.1. Let (X, α) be a properly based one-ended space in $\mathcal{P}$. We define the *proper m-dimensional L–S category* of X under α as the smallest number $p - \mathrm{cat}^m_{[\alpha]}(X) = n$ of elements in an open cover $\mathfrak{U} = \{U_1, \ldots, U_n\}$ of X such that for each $i \leqslant n$ and any proper map $f : K^s \to \overline{U}_i$ from any s-dimensional CW-complex ($s \leqslant m$) we have a diagram in $\mathcal{P}$

$$\begin{tikzcd} K^s \arrow[r, "r"] \arrow[d, "f"'] & \mathbb{R}_+ \arrow[d, "\alpha"] \\ \overline{U}_i \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

which is commutative up to proper homotopy; compare with [20] for the definition in ordinary homotopy theory. We call the cover $\mathfrak{U}$ above a (*proper*) *m-categorical cover* of X under α.

It is obvious that $p - \mathrm{cat}^m_{[\alpha]}(X) \leqslant p - \mathrm{cat}_{[\alpha]}(X)$. Next proposition provides us with a sufficient condition for the equality.

Proposition 5.2. *Let (X, α) be a properly based n-dimensional one-ended polyhedron in $\mathcal{P}$. Assume that the inequality $n - m + 1 \leqslant p - \mathrm{cat}^m_{[\alpha]}(X)$ holds. Then $p - \mathrm{cat}^m_{[\alpha]}(X) = p - \mathrm{cat}_{[\alpha]}(X)$.*

As a consequence of Proposition 5.2 we get

Corollary 5.3. *Let (X, α) be a properly based n-dimensional one-ended polyhedron in $\mathcal{P}$ then $p - \mathrm{cat}_{[\alpha]}(X) \leqslant \max\{2, p - \mathrm{cat}^{n-1}_{[\alpha]}(X)\}$.*

Proof. If $p - \mathrm{cat}^{n-1}_{[\alpha]}(X) = 1$ then the $(n-1)$-skeleton $X^{n-1} \subset X$ can be deformed to the ray α and hence X is properly $(n-1)$-connected. Thus $X \simeq S^n_\gamma$ has the same proper homotopy type as an n-spherical object by Proposition A.3 and so $p - \mathrm{cat}_{[\alpha]}(X) \leqslant 2$. □

Proposition 5.2 is a consequence of the following variation for m-dimensional category of Proposition 3.4 in [11].

Proposition 5.4. *Let (X,α) be a properly based n-dimensional one-ended polyhedron in $\mathcal{P}$ with $p-\mathrm{cat}^m_{[\alpha]}(X)=k$. Then there is an m-categorical cover $\{W_1,\dots,W_k\}$ of X under α such that each $W_i \searrow_p N_i$ where N_i is a polyhedron of dimension $\leqslant n-k+1$.*

The proof of Proposition 5.4 is similar to that of Proposition 3.4 in [11] based on arguments from PL topology. Recall that, given two polyhedra X and Y in $\mathcal{P}$ it is said that there is an elementary proper collapse from Y onto X if $Y=X\cup C_1\cup C_2\cup\cdots\cup C_n\cup\cdots$ where $\{C_i\}$ is a sequence of compact polyhedra satisfying $(C_i-X)\cap(C_j-X)=\emptyset$ if $i\neq j$ and each C_i collapses to $C_i\cap X$. Then a proper collapse $Y\searrow_p X$ is a finite sequence of elementary proper collapses.

Proof of Proposition 5.2. Let $p-\mathrm{cat}^m_{[\alpha]}(X)=k$. Then by Proposition 5.4 we obtain an m-categorical cover $\mathfrak{U}=\{W_i\}_{i\leqslant k}$ with $W_i\searrow_p N_i$ and $\dim N_i\leqslant n-k+1\leqslant m$. Hence each $N_i\subset W_i$ is properly deformable in X to the ray α. As W_i properly collapses to N_i, it follows that $\mathfrak{U}$ is actually a categorical cover of X, and so $p-\mathrm{cat}_{[\alpha]}(X)\leqslant p-\mathrm{cat}^m_{[\alpha]}(X)$. □

Next we prove a domination property for the proper 1-dimensional category; compare with Lemma 3.6.

Proposition 5.5. *Let (X,α) and (Y,β) be properly based one-ended spaces in $\mathcal{P}$. Assume that there exist proper maps $f:X\to Y$ and $g:Y\to X$ such that $f\alpha\simeq\beta$ and $g\beta\simeq\alpha$ and the induced pro-morphism $g_*f_*=\mathrm{id}:\mathrm{pro}-\pi_1(X,\alpha)\to\mathrm{pro}-\pi_1(X,\alpha)$ is the identity. Then $p-\mathrm{cat}^1_{[\alpha]}(X)\leqslant p-\mathrm{cat}^1_{[\beta]}(Y)$.*

Proof. Assume for a moment that the composite

$$g_*f_*:\left[K^1,X\right]\to\left[K^1,X\right] \tag{$*$}$$

is the identity for any 1-complex K^1 in $\mathcal{P}$ and let U be a 1-categorical closed subset of Y under β. Then $f^{-1}(U)$ is 1-categorical in X under α; indeed, for any proper map $h:K^1\to f^{-1}(U)$ we have a homotopy

$$h\simeq gfh:K^1\xrightarrow{fh}U\hookrightarrow Y\xrightarrow{g}X.$$

Here we use condition $(*)$ above. Moreover, fh is properly deformable to the ray β and so h is properly deformable to α.

It remains to show property $(*)$. We can assume that $K^1=S^1_\varepsilon$ is a 1-spherical object under some end-faithful tree T; moreover by the PHEP we can also assume that any proper map $h:S^1_\varepsilon\to X$ satisfies that the vertices in $T\subset S^1_\varepsilon$ are mapped into $\alpha(\mathbb{R}_+)$. Here we use that X is one-ended. Hence $h\cup\alpha:S^1_\varepsilon\cup\mathbb{R}_+\to X$ induces a proper map $\tilde{h}:\Sigma^1\to X$ where Σ^1 is the quotient space obtained by identifying each vertex $v\in T$ with $x_v\in\mathbb{R}_+$ if $h(v)=\alpha(x_v)$. This way, Σ^1 is in fact homotopy equivalent (under $\mathbb{R}_+$) to a one-ended 1-spherical object k: $S^1_\mu\simeq\Sigma^1$, and so by the hypothesis on g_*f_*, $gf\tilde{h}k\simeq\tilde{h}k$.

Hence, $gf\tilde{h} \simeq \tilde{h}$ since k is a homotopy equivalence, and so $gfh \simeq h$ by composing with the quotient map $q: S^1_\varepsilon \to \Sigma^1$. $\square$

Proposition 5.6. *Let (X^2, α) and (Y^2, β) be two properly based one-ended two-dimensional* CW*-complexes in $\mathcal{P}$ for which there exist pro-morphisms $\underline{\varphi}: \mathrm{pro} - \pi_1(X, \alpha) \to \mathrm{pro} - \pi_1(Y, \beta)$ and $\underline{\psi}: \mathrm{pro} - \pi_1(Y, \beta) \to \mathrm{pro} - \pi_1(X, \alpha)$ such that $\underline{\psi\varphi} = \mathrm{id}$ is the identity. Then $p - \mathrm{cat}_{[\alpha]}(X) \leqslant p - \mathrm{cat}_{[\beta]}(Y)$ unless $Y \simeq \mathbb{R}_+$ in which case $X \simeq S^2_\gamma$ is necessarily a 2-spherical object.*

Proof. According to Lemma 3.7 we can assume that α and β are cofibrations. Moreover, there are homotopy equivalences (under $\mathbb{R}_+$) $X^2 \simeq P^2$ and $Y^2 \simeq Q^2$ where P^2 and Q^2 are normalized CW-complexes; so we may assume $X^2 = P^2$ and $Y^2 = Q^2$. Then we can realize the pro-morphisms $\underline{\varphi}$ and $\underline{\psi}$ by proper maps f and g as follows; see Remark A.2. We start with the diagram

$$\underline{L}_2 = \mathrm{pro} - \pi_1\big(S^1_{\varepsilon_2}, \alpha\big) \xrightarrow{d_*} \underline{L}_1 = \mathrm{pro} - \pi_1\big(S^1_{\varepsilon_1}, \alpha\big) \overset{i_*}{\twoheadrightarrow} \underline{G} = \mathrm{pro} - \pi_1\big(X^2, \alpha\big)$$

induced by the attaching map of the 2-cells $d: S^1_{\varepsilon_2} \to S^1_{\varepsilon_1}$. Moreover $\underline{G} = \underline{L}_1/N(d_*(\underline{L}_2))$, where $N(\underline{H})$ denotes the tower obtained by normalizing levelwise the groups in $\underline{H}$. Then we realize $\underline{\varphi} i_*$ by a proper map $f_1: S^1_{\varepsilon_1} \to Y$ (Remark A.2) and since $i_* d_*$ is the trivial map then $f_1 d: S^1_{\varepsilon_2} \to Y$ is trivial in $\mathrm{pro} - \pi_1(Y, \beta)$ and hence extends to a proper map $\tilde{f}_2: B^2_{\varepsilon_2} \to Y$ from the corresponding "string" of disks into Y. As $X^2 = C_{\mathbb{R}_+} d$ is the mapping cone of d it follows that $\tilde{f}_2$ induces a proper map $f: X \to Y$ with $f_* = \underline{\varphi}$. Similarly we obtain g with $g_* = \underline{\psi}$. Now the result follows from Propositions 5.2 and 5.5 unless $Y \simeq \mathbb{R}_+$; here we use the fact that CW-complexes can be replaced by polyhedra up to proper homotopy equivalences. In case $Y \simeq \mathbb{R}_+$ it follows that $\mathrm{pro} - \pi_1(X, \alpha) = 0$ and then $X^2 \simeq S^2_\gamma$ is a 2-spherical object by Proposition A.3. $\square$

As a consequence of Propositions 4.6 and 5.6 we readily get

Corollary 5.7. *Let (X^2, α) be a properly based one-ended two-dimensional* CW*-complex in $\mathcal{P}$. Then $p - \mathrm{cat}_{[\alpha]}(X) \leqslant 2$ if and only if $\mathrm{pro} - \pi_1(X, \alpha)$ is pro-isomorphic to a tower $\underline{L} \vee \underline{P}$.*

Corollary 5.8. *The class of towers which are (pro-isomorphic to) coproducts $\underline{L} \vee \underline{P}$ of a free tower and a telescopic tower is closed under retracts in the subcategory $(\mathcal{G}r, \mathrm{tow} - \mathcal{G}r)_{f.p.} \subset (\mathcal{G}r, \mathrm{tow} - \mathcal{G}r)$ of finitely presented towers.*

By a *finitely presented* tower we mean a tower $\underline{G}$ in $(\mathcal{G}r, \mathrm{tow} - \mathcal{G}r)$ which is pro-isomorphic to a quotient tower $\underline{G} \cong \underline{L}_0/N(\underline{\varphi}(\underline{L}_1))$ where $\underline{\varphi}: \underline{L}_1 \to \underline{L}_0$ is a pro-morphism between free towers and $N(\underline{H})$ denotes the tower obtained by normalizing levelwise the groups in $\underline{H}$; compare [16].

Proof of Proposition 5.8. Consider a retraction diagram

$$\underline{G} \xrightarrow{i} \underline{L} \vee \underline{P} \xrightarrow{r} \underline{G}$$

where $\underline{G} \simeq \underline{L}_0/N(\underline{\varphi}(\underline{L}_1))$ is a finitely presented tower. Then we construct the properly based 2-dimensional polyhedron $(B(\underline{L} \vee \underline{P}), \beta)$ as the proper wedge of a one-ended 1-spherical object S^1_ε with pro $-\,\pi_1(S^1_\varepsilon, \beta) \cong \underline{L}$ (here $\beta : \mathbb{R}_+ \rightarrowtail S^1_\varepsilon$ is the canonical inclusion), and a proper wedge C of a decreasing sequence (possibly infinite) of cylinders $C_n = S^1 \times [n, \infty)$ and/or Euclidean planes $\mathbb{R}^2_m = S^1 \times [m, \infty)/S^1 \times \{m\}$ attached along the half line $\mathbb{R}_+$ for which pro $-\,\pi_1(C, \beta) \cong \underline{P}$ with $\beta : \mathbb{R}_+ \rightarrowtail C$ the canonical inclusion. Next, we realize the pro-morphism $\underline{\varphi} : \underline{L}_1 \to \underline{L}_0$ by a proper map $f : S^1_{\delta_2} \to S^1_{\delta_1}$ where $S^1_{\delta_i}$ are 1-spherical objects under $\mathbb{R}_+$ with pro $-\,\pi_1(S^1_{\delta_i}, \alpha) \cong \underline{L}_i$; see Remark A.2. Then the mapping cone $X^2 = C_{\mathbb{R}_+} f$ satisfies pro $-\,\pi_1(X^2, \alpha) \cong \underline{G}$. By Proposition 5.6, $p - \mathrm{cat}_{[\alpha]}(X^2) \leqslant p - \mathrm{cat}_{[\beta]}(B(\underline{L} \vee \underline{P})) \leqslant 2$ and, from Corollary 5.7, we get that $\underline{G}$ is a tower of the form $\underline{L} \vee \underline{P}$. □

6. Proper 1-types and 2-dimensional proper co-H-spaces

In ordinary homotopy theory, groups can be regarded as J.H.C. Whitehead's 1-types. Recall that two CW-complexes X and Y have the same *n-type* if there exist maps $f : X^{n+1} \to Y^{n+1}$ and $g : Y^{n+1} \to X^{n+1}$ such that the restrictions $gf|X^n$ and $fg|Y^n$ are homotopic to the inclusions $X^n \subseteq X^{n+1}$ and $Y^n \subseteq Y^{n+1}$ respectively. Notice that, by the cellular approximation theorem, the skeleton X^{n+1} has the same n-type as X. It is a classical result that two connected CW-complexes have the same 1-type if and only if they have isomorphic fundamental groups. More generally, we have the following result due to Whitehead.

Theorem 6.1 [27, Theorem 14]. *Let X^n and Y^n be two n-dimensional finite connected* CW-*complexes. Then they have the same $(n-1)$-type if and only if there exist finite one point union of spheres $\bigvee_{\alpha \in A} S^n_\alpha$ and $\bigvee_{\beta \in B} S^n_\beta$ such that $X^n \vee (\bigvee_{\alpha \in A} S^n_\alpha) \simeq Y^n \vee (\bigvee_{\beta \in B} S^n_\beta)$ are homotopy equivalent.*

Hence, finitely generated free groups represent in ordinary homotopy the 1-types of CW-complexes with L–S category $\leqslant 2$; i.e., co-H-spaces.

In proper homotopy theory n-types can be defined by the obvious extension of the definition above; however, in order to avoid the intricate question of "base point" in proper homotopy, we will consider here only properly based n-types. More precisely, let (X, α) and (Y, β) be properly based one-ended CW-complexes. We say that they have the same *proper n-type under* $\mathbb{R}_+$ ($n \geqslant 1$) if there exist proper maps under $\mathbb{R}_+$ (called *n-equivalences*) $f : X^{n+1} \to Y^{n+1}$ and $g : Y^{n+1} \to X^{n+1}$ such that the restrictions $gf|X^n$ and $fg|Y^n$ are homotopic to the inclusions $X^n \subseteq X^{n+1}$ and $Y^n \subseteq Y^{n+1}$ relative to α and β, respectively. A general treatment of proper n-types requires the subtle notion of a proper groupoid given in [10]; see also [7].

The arguments in the proofs of Propositions 5.5 and 5.6 and the proper cellular approximation theorem [10] readily lead us to the following proposition; compare with Corollary 4.3.

Proposition 6.2. *Two one-ended finite dimensional locally finite* CW*-complexes* (X,α) *and* (Y,β) *have the same proper* 1*-type under* $\mathbb{R}_+$ *if and only if their fundamental pro-groups* pro $-\pi_1(X,\alpha)\cong$ pro $-\pi_1(Y,\beta)$ *are pro-isomorphic.*

In addition we have

Proposition 6.3. *If* (X,α) *and* (Y,β) *are properly based one-ended* 2*-dimensional locally finite* CW*-complexes with the same proper* 1*-type then there are spherical objects* S^2_γ, S^2_δ *and a proper homotopy equivalence* $X\vee S^2_\gamma\simeq Y\vee S^2_\delta$ *relative to the base ray.*

Proof. We can assume that X and Y are normalized CW-complexes; that is, they are reduced mapping cones of maps $h:S^1_{\alpha_2}\to X^1=S^1_{\alpha_1}$ and $h':S^1_{\beta_2}\to Y^1=S^1_{\beta_1}$ respectively. By hypothesis, there exist cellular maps $f:X\to Y$ and $g:Y\to X$ such that $gf|X^1\simeq i$: $X^1\subset X$ and $fg|Y^1\simeq j$: $Y^1\subset Y$. If $f_1:X^1\to Y^1$ and $g_1:Y^1\to X^1$ are the corresponding restrictions to 1-skeletons we form from the reduced mapping cylinders $M_{\mathbb{R}_+}f_1$ and $M_{\mathbb{R}_+}g_1$ the push-outs of the obvious inclusions

$$\begin{tikzcd} X^1\vee Y^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & M_{\mathbb{R}_+}f_1 \arrow[d] \\ M_{\mathbb{R}_+}g_1 \arrow[r] & T_1 \end{tikzcd} \qquad \begin{tikzcd} X^1\vee Y^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & X\vee Y^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{push}"] & X\vee Y \arrow[d] \\ T_1 \arrow[r] & T' \arrow[r] & T \end{tikzcd}$$

The inclusion $Y^1\subset T_1$ is clearly homotopic to $g_1:Y^1\to X^1\subset T_1$ and so $h':S^1_{\beta_2}\to Y^1\subset T'$ is homotopic to the composite

$$S^1_{\beta_2}\xrightarrow{h'}Y^1\xrightarrow{g_1}X^1\subset X\subset T'$$

which is nullhomotopic since $g|Y^1$ extends to $Y=C_{\mathbb{R}_+}h'$ by g. Hence the gluing Lemma A.1 yields a homotopy equivalence $T\simeq T'\vee S^2_{\beta_2}$. Moreover, as $gf|X^1=g|Y^1f_1:X^1\to X$ is homotopic to the inclusion i, the homotopy equivalences (compare [14])

$$M_{\mathbb{R}_+}f_1\cup_{Y^1}M_{\mathbb{R}_+}g|Y^1\simeq M_{\mathbb{R}_+}g|Y^1f_1=M_{\mathbb{R}_+}gf|X^1\simeq M_{\mathbb{R}_+}i \text{ rel. } X^1\vee X$$

yield a homotopy equivalence $T'\simeq \Sigma_*X^1\cup_{X^1}X$. Here Σ_*X^1 is the *punctured torus* obtained by the lower push-out in the diagram

$$\begin{tikzcd} & X^1 \arrow[r] \arrow[d, "-i_1+i_0"'] \arrow[dl] \arrow[dr, phantom, "\text{push}"] & C_{\mathbb{R}_+}X^1 \arrow[d] \\ \mathbb{R}_+ \arrow[dr] & X^1\vee X^1 \arrow[r] \arrow[d, "{(1,1)}"'] \arrow[dr, phantom, "\text{push}"] & I_{\mathbb{R}_+}X^1 \arrow[d] \\ & X^1 \arrow[r] & \Sigma_*X^1 \end{tikzcd}$$

where the left-hand side triangle is commutative up to homotopy. This shows that $\Sigma_*X^1\simeq X^1\vee S^2_{\alpha_1}$ by the gluing Lemma A.1. Notice that the upper push-out follows from the co-

H-structure of $X^1 = S^1_{\alpha_1}$ which is a spherical object. Therefore we get

$$T \simeq T' \vee S^2_{\beta_2} \simeq X \vee S^2_{\beta_2} \vee S^2_{\alpha_1}.$$

Similarly $T \simeq Y \vee S^2_{\alpha_2} \vee S^2_{\beta_1}$. $\square$

Summarizing the results above we characterize proper co-H-spaces in dimension 2 in various ways. Namely,

Corollary 6.4. *Let (X,α) be a properly based one-ended two-dimensional* CW*-complex in $\mathcal{P}$. Then the following statements are equivalent*:

(1) *(X,α) is a proper co-H-space.*
(2) pro $-\,\pi_1(X,\alpha)$ *is pro-isomorphic to a tower of the form $\underline{L} \vee \underline{P}$.*
(3) $p-\mathrm{cat}_{[\alpha]}(X) \leqslant 2$.
(4) *There exist spherical objects S^2_α and S^2_β an a proper homotopy equivalence* (*under* $\mathbb{R}_+$) $X \vee S^2_\alpha \simeq B(\underline{L} \vee \underline{P}) \vee S^2_\beta$.

In fact, we have (1) $\Leftrightarrow$ (3) by 4.1, (2) $\Leftrightarrow$ (3) is 5.7 and (2) $\Leftrightarrow$ (4) follows from Proposition 6.3.

Remark 6.5. Proposition 6.3 is the proper analogue of the two-dimensional case of Whitehead's Theorem 6.1. There is a proper version of this theorem due to Zobel [28, Satz 6.11] which states that two one-ended n-dimensional CW-complexes in $\mathcal{P}$ (X^n,α) and (Y^n,β) have the same proper $(n-1)$-type under $\mathbb{R}_+$ if and only if there exist n-spherical objects S^n_ε and $S^n_{\varepsilon'}$ and a proper homotopy equivalence (under $\mathbb{R}_+$) $X^n \vee S^n_\varepsilon \simeq Y^n \vee S^n_{\varepsilon'}$. Compare with [9, II.6.1] where a proof of the classical Whitehead theorem is given by following the pattern of Zobel's proof in the proper setting. The alternative proof of Proposition 6.3 given here is purely homotopical. It relies crucially on the co-H-structure of 1-skeletons as spherical objects and seems not be appropriate for the general case.

With respect to Corollary 6.4, we conjecture that any two-dimensional proper co-H-space X is in fact proper homotopy equivalent to $B(\underline{L} \vee \underline{P}) \vee S^2_\beta$ for some spherical object S^2_β. This conjecture is the two-dimensional case of the proper analogue of a conjecture due to Ganea claiming that any finite CW-complex which is a co-H-space has the same homotopy type as a wedge of circles and a simply connected space. The following proposition supports our conjecture

Proposition 6.6. *If (X,α) is a properly based one-ended* 2*-dimensional locally finite* CW*-complex and* pro $-\,\pi_1(X,\alpha)$ *is pro-isomorphic to a tower of the form $\underline{L} \vee \underline{P}$, then there is a spherical object S^3_α and a proper homotopy equivalence $\Sigma X \simeq \Sigma B(\underline{L} \vee \underline{P}) \vee S^3_\alpha$.*

Proof. By Hurewicz theorem we get a pro-isomorphism pro $-\,H_1 X \cong$ pro $-\,H_1 B(\underline{L} \vee \underline{P}) = \underline{L}^{ab} \oplus \underline{P}^{ab}$, the latter being the abelianization of the tower $\underline{L} \vee \underline{P}$. This is a tower of free Abelian groups and hence it has projective dimension 1 (see [2]). Moreover, since X

is 2-dimensional and the kernel of a pro-morphism between free Abelian towers is also a free tower, we have that pro $-$ $H_2X = \underline{L}_\alpha$ is free.

Under these conditions ΣX admits a proper homology decomposition, see [1]. Furthermore, it is easy to check that $\Sigma B(\underline{L} \vee \underline{P})$ is a proper Moore space of type $(\underline{L}^{ab} \oplus \underline{P}^{ab}, 2)$, and S^2_α of type $(\underline{L}_\alpha, 2)$. Therefore, ΣX is homotopy equivalent to the proper mapping cone of a proper map $S^2_\alpha \to \Sigma B(\underline{L} \vee \underline{P})$ which is trivial in pro-homology, and hence by Hurewicz theorem and Remark A.2 it is null-homotopic, so we get the desired homotopy equivalence. □

Recently Iwase [22] has proved that Ganea's conjecture is false. Moreover, he gives sufficient conditions to get a positive answer to the conjecture; in particular he shows that the conjecture is true for finite two-dimensional CW-complexes; that is, any finite two-dimensional CW-complex which is a co-H-space is homotopy equivalent to a finite wedge $\bigvee_{\alpha\in A} S^1_\alpha \vee (\bigvee_{\beta\in B} S^2_\beta)$. An algebraic question closely related to our conjecture above is to know whether projective direct summands of free $U(\underline{L} \vee \underline{P})$-modules in the sense of [10], see also [18], are free. We will tackle these problems in a near future.

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Appendix A. A brief review of proper algebraic topology

Although the material collected here is well known for an advanced reader, with this appendix we intend to provide the non-specialists with a sketch of the basic facts of proper algebraic topology used in the paper. Recall that we work within the category $\mathcal{P}$ of locally path connected, locally compact σ-compact Hausdorff spaces and proper maps. In order to set up the homotopy theory of $\mathcal{P}$ one observes that this category is not closed under push-outs but it contains sufficiently many to allow the basic homotopical constructions. Actually the category $\mathcal{P}$ is a cofibration category in the sense of Baues [8,3]. The ordinary cylinder functor $IX = X \times I$, with inclusions $i_\varepsilon(x) = (x, \varepsilon)$ $(\varepsilon = 0, 1)$, and the proper cofibrations endow the category $\mathcal{P}$ of a structure of cofibration category, in fact an I-category; see [3]. In this structure the *mapping cylinder* Mf of a proper map $f : X \to Y$ is defined as the push-out of the diagram $IX \xleftarrow{i_0} X \xrightarrow{f} Y$ while the (*proper*) *cone* CX of a space X in $\mathcal{P}$ is the push-out $IX \xleftarrow{i_1} X \xrightarrow{r} \mathbb{R}_+$. More generally, the *mapping cone* Cf of a map $f : X \to Y$ in $\mathcal{P}$ is the push-out of the diagram $Y \xleftarrow{f} X \xrightarrow{k} CX$ where $k : X \to CX$ is the canonical embedding $x \mapsto [x, 0]$. The (*proper*) *suspension* ΣX of a space X in $\mathcal{P}$ is then the mapping cone Cr of any proper map $r : X \to \mathbb{R}_+$. Since the proper homotopy class of r is unique, the proper homotopy types of proper cones and suspensions are well defined by the following gluing lemma available in any cofibration category [8].

Lemma A.1. *Consider the commutative diagram in* $\mathcal{P}$

$$\begin{tikzcd} Y_0 \arrow[d, "\alpha"'] & Y \arrow[l] \arrow[r] \arrow[d, "\gamma"] & Y_1 \arrow[d, "\beta"] \\ X_0 & X \arrow[l] \arrow[r] & X_1 \end{tikzcd}$$

where at least one arrow in each row is a cofibration. Assume in addition that α, β, γ *are proper homotopy equivalences. Then the natural map* $\alpha \cup \beta : Y_0 \cup_Y Y_1 \to X_0 \cup_X X_1$ *between the corresponding push-outs is a proper homotopy equivalence. As a consequence, the proper homotopy type of* $X_0 \cup_X X_1$ *only depends on the proper homotopy classes of the maps involved in its definition.*

If (X, α) is a properly well based space in $\mathcal{P}$, the corresponding reduced version of cylinder, cone and suspension are obtained by identifying $\alpha(\mathbb{R}_+) \times I$ to $\alpha(\mathbb{R}_+)$ via the projection $I \to \{p\}$ to the one-point space; we denote them by $I_{\mathbb{R}_+}X$, $C_{\mathbb{R}+}X$ and $\Sigma_{\mathbb{R}_+}X$ and they are properly well based spaces in $\mathcal{P}$. Let $\mathcal{P}^{\mathbb{R}_+}$ denote the category of properly well based spaces in $\mathcal{P}$ with proper maps $f : (X, \alpha) \to (Y, \beta)$ with $f\alpha = \beta$. A *homotopy under* $\mathbb{R}_+$ between $f, g : (X, \alpha) \to (Y, \beta)$ is a map $H : (I_{\mathbb{R}_+}X; \alpha) \to (Y, \beta)$ in $\mathcal{P}^{\mathbb{R}_+}$. The reduced mapping cone and mapping cylinder of a map $f : X \to Y$ in $\mathcal{P}^{\mathbb{R}_+}$ will be denoted by $C_{\mathbb{R}_+} f$ and $M_{\mathbb{R}_+} f$ respectively.

The algebraic invariants in proper homotopy theory are given here by the usual language for "proper" algebra provided by towers of algebraic objects. Recall that given a category $\mathcal{C}$, the *category of towers* of $\mathcal{C}$, tow $-\, \mathcal{C}$, is the category of inverse sequences $\underline{A} = \{A_1 \leftarrow A_2 \leftarrow \cdots\}$ in $\mathcal{C}$ and pro-morphisms. See [17] for details about pro-categories. We will also use the full subcategory of $\mathcal{M}$or(tow $-\, \mathcal{C}$) whose objects are arrows $\underline{f} : \underline{X} \to A$ where $\underline{X}$ is a (tow $-\, \mathcal{C}$)-object and A is a $\mathcal{C}$-object regarded as a constant tower whose bonding maps are the identity. This category is denoted $(\mathcal{C}, \text{tow} - \mathcal{C})$. The object $\underline{f} : \underline{X} \to A$ can be represented as a tower $\{A \leftarrow X_{n_1} \leftarrow X_{n_2} \leftarrow \cdots\}$ for some subsequence $n_1 > n_2 > \cdots$ with A as a fixed object, and a morphism from $\underline{f} : \underline{X} \to A$ to $g : \underline{Y} \to B$ can be regarded as a $\mathcal{C}$-morphism between A and B together with a (tow $-\, \mathcal{C}$)-morphism from $\underline{X}$ to $\underline{Y}$ such that both morphisms are compatible via the bonding maps. Morphisms in $(\mathcal{C}, \text{tow} - \mathcal{C})$ will be also called *pro-morphisms*.

Let $\mathcal{C} = \mathcal{G}r$ be the category of groups, given a properly based space (X, α) in $\mathcal{P}$, the *nth homotopy pro-group* of (X, α) is the object in $(\mathcal{G}r, \text{tow} - \mathcal{G}r)$

$$\text{pro} - \pi_n(X, \alpha) = \big\{\pi_n(X, x_0) \leftarrow \pi_n(U_1, x_1) \leftarrow \pi_n(U_2, x_2) \leftarrow \cdots\big\}$$

where $\{U_j\}$ is a system of ∞-neighbourhoods, $x_j = \alpha(t_j)$ with $\alpha([t_j, \infty)) \subseteq U_j$, and the bonding morphisms are induced by inclusions and base-point change isomorphisms along the base ray α. When $n = 1$ pro $-\, \pi_1(X, \alpha)$ is called the *fundamental pro-group* of (X, α).

Let $\mathcal{C} = \mathcal{A}b$ be the category of Abelian groups, given a space X in $\mathcal{P}$, the n*th homology pro-group* of X is defined as the object in $(\mathcal{A}b, \text{tow} - \mathcal{A}b)$

$$\text{pro} - H_n(X) = \big\{H_n(X) \leftarrow H_n(U_1) \leftarrow H_n(U_2) \leftarrow \cdots\big\}.$$

It is can be checked that for a CW-complex in $\mathcal{P}$ the homology tower pro $- H_n(X)$ can be also obtained as the homology of the chain complex in $(\mathcal{A}b, \mathrm{tow} - \mathcal{A}b)$

$$\mathrm{pro} - C_n(X) = \{C_n(X) \leftarrow C_n(U_1) \leftarrow C_n(U_2) \leftarrow \cdots\}$$

where $\{U_j\}$ is a system of neighbourhoods at infinity consisting of CW-complexes and $C_n(U_j)$ is the free Abelian group of cellular n-chains of U_j.

For the transition from topology to algebra, free towers play a crucial role. Here, by a *free tower* in $(\mathcal{G}r, \mathrm{tow} - \mathcal{G}r)$ we mean a tower of the form

$$\underline{L} = \{L_0 \xleftarrow{i_1} L_1 \xleftarrow{i_2} \cdots\} \tag{A.1}$$

where $L_i = \langle B_i \rangle$ are free groups of basis B_i such that $B_{i+1} \subset B_i$, the differences $B_i - B_{i+1}$ are finite and $\bigcap_{i=0}^{\infty} B_i = \emptyset$. Finally the bonding homomorphisms i_k are induced by the corresponding basis inclusions. Similarly one can define free towers in $(\mathcal{A}b, \mathrm{tow} - \mathcal{A}b)$, an example of such towers is the tower of cellular n-chains pro $- C_n(X)$ of a CW-complex in $\mathcal{P}$.

Observe that the free tower $\underline{L}$ is completely determined by the proper map $\alpha : B_0 \to \mathbb{N}$ from the discrete set B_0 to the 0-skeleton $\mathbb{N} \subset \mathbb{R}_+$. Namely, $\alpha(b) = n$ if $b \in B_n - B_{n+1}$ and hence $B_n = \{\alpha^{-1}(m);\ m \geqslant n\}$. If we want to make explicit the "height" function α we write $\underline{L} = \underline{L}_\alpha$. Moreover, by use of the fundamental pro-group we can identify the free tower $\underline{L}_\alpha$ with the space S^1_α obtained by attaching $\#\alpha^{-1}(n)$ 1-spheres at each vertex $n \in \mathbb{R}_+$. The space S^1_α is termed a 1-*spherical object* under $\mathbb{R}_+$. The 2-disk B^2 yields the corresponding object B^2_α. Similarly one can define spherical objects under an arbitrary tree T in $\mathcal{P}$ as well as higher dimensional spherical objects S^n_α.

Remark A.2. The identification of the spherical object S^n_α with the free tower $\underline{L}_\alpha$ of groups (Abelian groups if $n \geqslant 2$) via the pro-isomorphism pro $- \pi_n(S^n_\alpha) \cong \underline{L}_\alpha$ allows us to realize for a properly based space (X, β) any pro-morphism in $(\mathcal{C}, \mathrm{tow} - \mathcal{C})((L_\alpha, \mathrm{pro} - \pi_n(X, \beta))$ $(\mathcal{C} = \mathcal{G}r$ or $\mathcal{A}b)$ by a proper map $f : S^n_\alpha \to X$ (under $\mathbb{R}_+$). More explicitly, there is a 1–1 correspondence $(\mathcal{C}, \mathrm{tow} - \mathcal{C})(L_\alpha, \mathrm{pro} - \pi_n(X, \beta)) \cong [S^n_\alpha, X]$ which carries the proper homotopy class $[f]$ under $\mathbb{R}_+$ to the induced pro-morphism $f_* : \mathrm{pro} - \pi_n(S^n_\alpha) \to \mathrm{pro} - \pi_n(X, \beta)$.

A space X in $\mathcal{P}^{\mathbb{R}_+}$ is said to be *properly k-connected* if it is one-ended and pro $- \pi_n(X, \alpha) = 0$ for $n \leqslant k$. The homotopy category of n-dimensional properly $(n-1)$-connected CW-complexes coincides with the homotopy category of n-spherical objects. More explicitly,

Proposition A.3. *Let X be a properly $(n-1)$-connected locally finite* CW-*complex of dimension $n \geqslant 1$. Then X has the proper homotopy type of an n-spherical object.*

This proposition follows from the proper homological Whitehead theorem (see [17] or [10]) and the interesting fact that the category of free towers of Abelian groups admits kernels; see [2] for an elementary proof.

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